*Geophys. J. R. astr. Soc.* (1984) 78, 105–118

# Numerical simulation of attenuated wavefields using a Padé approximant method

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Received 1983 October 12

**Summary.** Realistic anelastic attenuation laws are usually formulated as convolution operators, but this representation is intractable for time-domain synthetic seismogram methods such as the finite difference method. An approach based on Padé approximants provides a convenient, accurate reformulation of general anelastic laws in differential form. The resulting differential operators form a uniformly convergent sequence of increasing order in the time derivative, and all are shown to be causal, stable and dissipative. In the special case of frequency-independent $Q$, all required coefficients for the operators are obtained in closed form in terms of Legendre polynomials.

Low-order approximants are surprisingly accurate. Finite-difference impulse responses for a plane wave in a constant-$Q$ medium, calculated with the fifth-order convergent, are virtually indistinguishable from the exact solution. The formulation is easily generalized to non-scalar waves. Moreover, this method provides a framework for incorporating amplitude-dependent attenuation into numerical simulations.

## 1 Introduction

This paper describes an efficient scheme for incorporating general anelastic attenuation laws into time-marching methods for computing theoretical seismograms. Time-marching refers to those methods which compute seismic time histories by direct numerical integration, in time, of the equations of motion. Examples of some time-marching methods widely used in seismology are: (1) the finite difference method (e.g. Boore 1972), (2) the finite element method (e.g. Smith 1975), (3) the Fourier, or pseudospectral, method (e.g. Kosloff & Baysal 1982), in which spatial derivatives are computed by means of the fast Fourier transform instead of finite differences, and (4) the discrete wavenumber finite element (DWFE) and related methods (Alekseev & Mikhailenko 1980; Olson, Orcutt & Frazier

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1984), in which horizontal dependence of the wavefield is Fourier–Bessel transformed before derivatives with respect to the vertical coordinate are replaced by finite differences. The class of time-marching methods excludes methods which construct theoretical time histories by Fourier synthesis of steady-state solutions. The latter includes, for example, the methods of reflectivity (e.g. Fuchs & Müller 1971), modal synthesis (e.g. Swanger & Boore 1978), and real-axis wavenumber integration (e.g. Apsel 1979). The approximate approach described herein is not required for the Fourier synthesis methods, since completely general linear attenuation laws can be incorporated into frequency domain formulations exactly, simply by making the elastic moduli complex functions of frequency.

In the time domain, on the other hand, the most general linear constitutive relationship expresses the stress at time $t$ as a hereditary integral of the strain over all earlier time. This formulation, while fully general, is completely useless for wave-propagation calculations. Immense storage and arithmetic requirements render the requisite numerical integrations intractable. As a result, wave propagation calculations by time-marching methods have generally been restricted to perfect elasticity and a few simple spring–dashpot models such as the Voigt solid.

Seismic data suggest that the internal friction of rock, $Q^{-1}$, is nearly independent of period, at least over the more than three decades from about 1 s to 1 hr. This observation requires that anelastic effects be characterized by a broad relaxation spectrum (Liu, Anderson & Kanamori 1976). The elementary spring–dashpot models are highly unrealistic in this respect. Lack of generality in representing anelastic attenuation is therefore a significant disadvantage of the existing time-marching methods relative to Fourier synthesis methods, for seismic applications.

The approach adopted here is to transform the convolution integral relating stress to strain history into a convergent sequence of constant-coefficient differential operators of increasing order. This is accomplished using the method of Padé approximants. Partial fraction expansion of the $n$th-order approximant then yields a first-order differential equation for each of $n$ internal variables, which sum to form the macroscopic stress. These first-order equations are easily approximated by finite differences for use in wave propagation calculations. The internal variables must be stored, so the storage requirement for finite difference calculations, for example, is $n$ times the number of stress components (1 for acoustic or *SH*-waves, 3 for 2-D plane strain, etc.) times the number of mesh points. In practice, fairly low-order approximants have been found to give extremely accurate results in wave-propagation calculations.

This particular approximation method appears to be very well suited to numerical calculations. In the most important special case, that of constant $Q$, the operator coefficients of any order can be obtained in closed form in terms of the Legendre polynomials. Even in the most general case, the coefficients can be automatically generated without recourse to curve-fitting. Moreover, each differential operator is guaranteed to be causal, stable and dissipative, and uniform convergence of the sequence is assured.

Finite difference calculations of plane-wave propagation in a constant $Q$ medium verify the stability and accuracy of the method. Using the fifth-order approximant, computed waveforms for the constant $Q$ medium are indistinguishable from an exact solution obtained by Fourier synthesis.

The formulation is easily generalized to non-scalar waves. It provides a solution to a long-standing problem in seismogram synthesis, with obvious applications to forward modelling in both strong motion and exploration seismology. Moreover, this method provides a framework for incorporating amplitude-dependent attenuation (e.g. Mavko 1979) into numerical simulations.

## 2 Background

The general linear, shift-invariant, isotropic stress–strain relationship can be resolved into separate 1-D relations of the form

$$\sigma(t) = \int_0^t M(t-t')\, d\epsilon(t'), \tag{1}$$

where $\sigma$ is a stress and $\epsilon$ a strain. The function $M$ is the stress history corresponding to a unit step function strain history. In terms of $M$, we can define the unrelaxed modulus, $M_u$, the relaxed modulus $M_R$, the relaxation of the modulus, $\delta M$, and the normalized relaxation function, $\phi$, by means of the following expressions:

$$M_u = M(0), \tag{2}$$

$$M_R = M(\infty), \tag{3}$$

$$\delta M = M_u - M_R, \tag{4}$$

and

$$M(t) = M_R + \delta M \phi(t), \tag{5}$$

so that $\phi(0) = 1$ and $\phi(\infty) = 0$. We will restrict attention to anelastic material response, which requires $M_R > 0$.

In what follows, we assume that the relaxation function $\phi$ has the following representation:

$$\phi(t) = \int_{-\infty}^{\infty} \Phi(\ln \tau) \exp(-t/\tau)\, d(\ln \tau), \tag{6}$$

with $\Phi \geq 0$. The non-negative distribution $\Phi(\ln \tau)$ is called the normalized relaxation spectrum. Nowick & Berry (1972) summarize the physical basis for representing $\phi$ in the form (6).

We apply the Laplace transform in $s$-multiplied form, i.e.

$$\bar{f}(s) = s \int_0^{\infty} f(t) \exp(-st)\, dt, \tag{7}$$

to (5) and (6). With the change of variable $p = \tau^{-1}$, we obtain the following expression for the operational modulus $\bar{M}$, the Laplace transform of $M$:

$$\bar{M}(s) = M_u - \delta M \int_0^{\infty} \frac{\hat{\Phi}(p)\, dp}{s+p}. \tag{8}$$

The distribution $\hat{\Phi}(p)$ is the relaxation spectrum defined by

$$\hat{\Phi}(p) = \Phi(-\ln p). \tag{9}$$

Equation (1) transforms to

$$\bar{\sigma}(s) = \bar{M}(s)\, \bar{\epsilon}(s), \tag{10}$$

which is the operational form of the stress–strain relationship.

Given a relaxation spectrum $\Phi$, our objective is to construct stable, accurate approximations to (1) in differential form. This simply requires approximating $\bar{M}(s)$ by a rational function $\bar{M}_n(s)$, in which $n$ denotes the degree of the denominator polynomial. Then, with

the replacement of $\bar{M}$ by $\bar{M}_n$ and the substitution of $d/dt$ for $s$, (10) becomes an $n$th-order differential equation for $\sigma(t)$, which can be discretized and time stepped. However, it is essential that the rational approximation $\bar{M}_n$ be of relatively low order, since storage and arithmetical requirements will be proportional to $n$.

If $\bar{M}(s)$ were initially a rational function of $s$, with denominator polynomial of sufficiently low order, obviously no approximation would be necessary. This is just the case in which the given relaxation spectrum consists of a small number of delta functions. In this case, each spectral line corresponds to a pole of the operational modulus $\bar{M}(s)$, and (8) reduces to the partial fraction expansion of $\bar{M}$.

In general, however, $\bar{M}(s)$ is not rational. For example, if $\Phi$ is a non-zero and continuous function of $\ln \tau$ over any finite interval, it can be deduced from the form (8) that $\bar{M}(s)$ has a branch cut on the negative real axis (e.g. Van der Pol & Bremmer 1964, pp. 77–78). The approximation method presented in the next section can be viewed as a systematic scheme for replacement of the branch cut for $\bar{M}(s)$ by a series of interlaced poles and zeros. The method is notable for retaining high accuracy at wavefronts with relatively few poles, $n$. Moreover, the resultant stress history converges everywhere (not just at wavefronts) to the exact solution, as $n$ increases.

As a practical matter, moreover, even if we initially have a pole-zero representation of $\bar{M}(s)$, but with a large number of poles, $N$, the approximation method replaces it by a reduced pole-zero representation. In this case, the $N$th-order approximant reproduces $\bar{M}$ exactly; but for $n < N$, the poles of $\bar{M}_n$ will not generally coincide with any of those of $\bar{M}$.

## 3 Padé approximants for the operational modulus

In this section we develop the method for approximating the operational modulus $\bar{M}$ by a rational fraction $\bar{M}_n$, given a relaxation spectrum $\Phi$. Both $\bar{M}(s)$ and $\phi(t)$ can be expanded in terms of the moments of the relaxation spectrum $\hat{\Phi}$. Examination of these expansions motivates the use of an approximation for $\bar{M}$ which preserves the maximal number of low-order relaxation moments. Then the theory of Padé approximants for Stieltjes series facilitates the construction of explicit expressions for $\bar{M}_n$ in terms of $\hat{\Phi}$, and the same theory provides us with convergence and stability proofs. Direct numerical methods for generating the approximants have been bypassed by using some classical results relating diagonals of the Padé table to sequences of orthogonal polynomials. This connection to orthogonal polynomials leads to a completely algebraic solution for the constant-$Q$ operator, which is given in the next section.

### 3.1 APPROXIMATION METHOD

As a preliminary step we rewrite (8) as:

$$\bar{M}(s) = M_u - \delta M s^{-1} \chi(-s^{-1}), \tag{11}$$

where we have introduced the new function

$$\chi(z) = \int_0^\infty \frac{\hat{\Phi}(p)\, dp}{1 - pz}. \tag{12}$$

The function $\chi$ is closely related to the Laplace-transform of relaxation function $\bar{\phi}$:

$$\chi(-s^{-1}) = s[1 - \bar{\phi}(s)]. \tag{13}$$

The reason for introducing $\chi$ is that it is the generating function for the moments of $\hat{\Phi}$: under repeated differentiation of the integral, (12) becomes

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{14}$$

with,

$$c_n = \int_0^{\infty} p^n \hat{\Phi}(p)\, dp, \qquad \hat{\Phi}(p) \geq 0. \tag{15}$$

Equations (14) and (15) define a Stieltjes series, i.e. a power series with coefficients equal to successive moments of a non-negative distribution (irrespective of whether the series converges; e.g. Baker 1975, chapters 15–17). This fact will be exploited extensively in what follows, enabling us to develop approximations to $\chi$ which match the low-order relaxation moments, yet converge even outside the radius of convergence of (14).

We also note that if (14) is substituted into (13), the resulting expression can be Laplace inverted term by term to give the MacLaurin series of the relaxation function $\phi(t)$ in terms of the moments. The result is

$$\phi(t) = 1 + \sum_{k=1}^{\infty} \frac{c_k (-t)^{k+1}}{(k+1)!}. \tag{16}$$

The important point for our purpose is that in so far as an approximation to $\chi$ preserves the moments of $\hat{\Phi}$, it will preserve the MacLaurin series of $\phi(t)$.

We now proceed to approximate the operational modulus by replacing $\chi(z)$ with a rational function $A_m(z)/B_n(z)$, where $A_m$ and $B_n$ are polynomials of degree $m$ and $n$, respectively. Two considerations require mention at this point. First we must set the numerator degree $m$ equal to $n-1$. This is required in order that the instantaneous elastic modulus $M_u$ be preserved by the approximation and that the late-time response remain bounded. Second, we note that in order to simulate accurately seismic amplitudes with a differential operator of low order, it is desirable that the corresponding approximation to the relaxation function be highly accurate at early time, i.e. at wavefronts.

These considerations lead us to choose $A_{n-1}$ and $B_n$ so as to match the relaxation moments $c_k$ up to the highest power possible for the given denominator degree $n$. We form the approximation $\chi_n$ given by

$$\chi_n(z) = \frac{A_{n-1}(z)}{B_n(z)} \tag{17}$$

subject to the constraint

$$\chi(z)\, B_n(z) - A_{n-1}(z) = 0(z^{2n}). \tag{18}$$

If we add the normalization

$$B_n(0) = 1, \tag{19}$$

and the stipulation that $A_{n-1}$ and $B_n$ have no common factors, then $\chi_n$ is uniquely determined (Baker 1975, p. 8), and is, by definition, a special case of the Padé approximant to $\chi$ (the $[n-1/n]$ Padé approximant, in the notation of Baker). Since $\chi$ is a Stieltjes series, the existence of $\chi_n$ is assured for all $n$ (Baker & Graves-Morris 1981, p. 164). (This is strictly true only when the support of the distribution $\hat{\Phi}$ is an infinite set. If, on the other hand,

$\hat{\Phi}$ consists only of a string of $N$ delta functions, then $\chi_n$ exists for $n \leq N$ and $\chi_N = \chi$.) Through (18) the approximation preserves the first $2n$ relaxation moments ($c_0$ to $c_{2n-1}$) and therefore (by 16) the first $2n$ derivatives of the relaxation function $\phi(t)$ at $t = 0^+$. As a result, we will find that even very low-order Padé approximants give highly accurate results in wave-propagation applications.

### 3.2 EXPLICIT EXPRESSIONS FOR THE APPROXIMANTS

Since $\chi_n$ is a Padé approximant to a Stieltjes series, we now have at our disposal a well-developed theory for constructing $\bar{M}_n$ for arbitrary $n$, as well as for assessing the convergence of the sequence $\bar{M}_n$. There are several purely numerical ways to proceed, but the most fruitful approach for our purposes is via the relationship between the Padé approximants and orthogonal polynomials. This approach leads to a closed form expression for $\bar{M}_n$ in the most important special case for seismic applications; in the most general case, this approach leads to a convenient, stable constructive algorithm.

We assume for simplicity that $\hat{\Phi}$ has a bounded interval of support $[\tau_2^{-1}, \tau_1^{-1}]$, with $\tau_2 > \tau_1 > 0$. Since $\hat{\Phi}$ is non-negative, we can define a set of polynomials $p_l(x)$, orthonormal with respect to $\hat{\Phi}$, i.e.

$$\int_{\tau_2^{-1}}^{\tau_1^{-1}} p_l(x)\, p_m(x)\, \hat{\Phi}(x)\, dx = \delta_{lm}, \qquad l, m = 0, 1, 2, \ldots \tag{20}$$

With the added condition that $p_m$ is of degree $m$, and that its leading coefficient $k_m$ is real and positive, the set is unique. (If $\hat{\Phi}$ consists solely of $N$ delta functions, then $p_m$ only exists for $m = 0, 1 \ldots, N-1$, and is called a discrete orthogonal polynomial. The subsequent development still applies for $n < N-1$, with exact convergence for $n = N$, as stated earlier.) Then $\chi_n$ is given by (e.g. Brezinski 1977)

$$\chi_n(z) = \frac{z^{-1}\, q_n(z^{-1})}{p_n(z^{-1})}, \tag{21}$$

where $q_n(x)$ is the associated polynomial (of degree $n-1$) given by

$$q_n(x) = \int_{\tau_2^{-1}}^{\tau_1^{-1}} \frac{p_n(t) - p_n(x)}{t - x}\, \hat{\Phi}(t)\, dt. \tag{22}$$

Using results summarized by Szegö (1939, p. 48) we readily obtain the partial fraction expansion of $\chi_n$,

$$\chi_n(z) = \sum_{i=1}^{n} \frac{\lambda_i}{1 - \nu_i z} \tag{23}$$

where $\nu_i$, $i = 1 \ldots n$ are the zeros of $p_n$. The residues, $\lambda_i$, are all positive, and are given by

$$\lambda_i = \frac{k_n}{k_{n-1}\, p_{n-1}(\nu_i)\, p_n'(\nu_i)}. \tag{24}$$

Note that the right-hand side of (23) is also a Gaussian quadrature formula for (12), with respect to weighting by $\hat{\Phi}$.

Substituting $\chi_n$ for $\chi$ in (11) gives the desired rational approximation for $\bar{M}$, denoted

by $\bar{M}_n$,

$$\bar{M}_n(s) = M_u - \delta M \sum_{i=1}^{n} \frac{\lambda_i}{s+\nu_i}, \tag{25}$$

which amounts to replacing $\hat{\Phi}$ by a string of positive delta functions located at the zeros of $p_n$. With $\bar{M}_n$ replacing $\bar{M}$, (10) implies the stress–strain relationship

$$\sigma(t) = M_u \left[ \epsilon(t) - \sum_{i=1}^{n} \zeta_i(t) \right], \tag{26}$$

where $\zeta_i$, $i = 1, \ldots n$, satisfy, respectively, the $n$ first-order differential equations

$$\frac{d\zeta_i(t)}{dt} + \nu_i \zeta_i(t) = \lambda_i \frac{\delta M}{M_u} \epsilon(t), \qquad i = 1, \ldots n. \tag{27}$$

The $\zeta_i$ can be viewed as internal variables which sum to give the macroscopic stress $\sigma$ (e.g. Minster 1980).

### 3.3 PROPERTIES

Equations (26) and (27) give the desired differential approximation to equation (1). Accuracy of the approximation at wavefronts was ensured by the method of construction. Stability of equation (27) requires $\mathrm{Re}\,[\nu_i] > 0$, and this is guaranteed by the fact that the zeros of orthogonal polynomials are always real and confined to the interval of orthogonality.

Furthermore, we can show that $\bar{M}_n$ is always dissipative. To do so, it suffices to show that $s\bar{M}_n^{-1}(s)$ is a positive real function (Brennan & Smylie 1981), i.e. $\mathrm{Re}\,[s\bar{M}_n^{-1}(s)] \geqslant 0$ whenever $\mathrm{Re}\,[s] \geqslant 0$ (and that $s\bar{M}^{-1}(s)$ is real when $s$ is, which is obvious). Given that the residues $\lambda_i$ are all positive, a straightforward calculation shows that the positive real property of $s\bar{M}^{-1}(s)$ follows if $\bar{M}_n(s)$ is itself positive real. But, $\mathrm{Re}\,[\bar{M}_n(s)]$, for $\mathrm{Re}\,[s] > 0$, is bounded below by

$$\mathrm{Re}\,[\bar{M}_n(s)] \geqslant M_u - \delta M \sum_{i=1}^{n} \frac{\lambda_i}{\nu_i}, \qquad \mathrm{Re}\,[s] \geqslant 0. \tag{28}$$

The sum is the Gaussian quadrature formula for

$$\int_{\tau_2^{-1}}^{\tau_1^{-1}} \frac{\hat{\Phi}(p)\,dp}{p} = 1$$

(the right hand side follows from the normalization $\phi(0) = 1$). The error formula for Gaussian quadrature is

$$\int_{\tau_2^{-1}}^{\tau_1^{-1}} f(x)\,\hat{\Phi}(x)\,dx - \sum_{i=1}^{n} \lambda_i f(\nu_i) = \frac{f^{(2n)}(\xi)}{(2n)!\,k_n^2}, \tag{29}$$

where $\xi$ is some point in the interval of integration (e.g. Hildebrand 1974, p. 388). Applying this formula to $1/p$ (whose even derivatives are positive), and noting again that the $\lambda_i$s are all positive, we see that the sum in (28) is a positive number less than 1. Therefore, we have

$$\mathrm{Re}\,[\bar{M}_n(s)] \geqslant M_u - \delta M = M_R > 0, \qquad \mathrm{Re}\,[s] \geqslant 0, \tag{30}$$

which proves that $\bar{M}_n$ is dissipative *for any choice of n.*

Finally, we can show that the sequence $\{M_n\}$ generates a convergent sequence of approximants, $\{Q_n^{-1}(\omega)\}$, to the internal friction $Q^{-1}(\omega)$. We use the definition of $Q^{-1}$ advocated by O'Connell & Budiansky (1978),

$$Q^{-1}(\omega) = \frac{\mathrm{Im}\,[\bar{M}(i\omega)]}{\mathrm{Re}\,[\bar{M}(i\omega)]}, \qquad \text{for real } \omega. \tag{31}$$

Since we have already shown that the real part of $\bar{M}_n(i\omega)$ cannot vanish for real $\omega$ (nor can the real part of $\bar{M}(i\omega)$), we have only to show that

$$\lim_{n\to\infty} \bar{M}_n(i\omega) = \bar{M}(i\omega).$$

This result follows from a convergence theorem for the Padé approximants to Stieltjes series (e.g. Baker 1970, p. 11), a sufficient condition in the present case being $\tau_1 > 0$. Thus, even though we constructed $\bar{M}_n$ to be particularly accurate at wavefronts, we have actually generated a convergent sequence, in the sense that we have ensured

$$\lim_{n\to\infty} Q_n^{-1}(\omega) = Q^{-1}(\omega) \tag{32}$$

for any frequency $\omega$.

In the Appendix we also establish the even stronger result that the sequence of differential operators defined by (26) and (27) converged in the uniform operator norm to the integral operator defined by (1). This result can be expressed in the following manner: by a suitable choice of $n$, independent of the strain history $\epsilon$, we can make the maximum error in the stress history arbitrarily small in any finite time interval.

### 3.4 COMPUTATION OF THE POLES AND RESIDUES

For constant $Q$ (uniform relaxation spectrum), the poles and zeros can be obtained in closed form in terms of tabulated quantities, as shown in the next section. For the most general distribution $\hat{\Phi}$, however, the orthogonal polynomials have to be represented numerically and their roots computed. A convenient and stable numerical representation of a set of orthogonal polynomials is in terms of its recursion coefficients, which are easily computed, either directly from $\hat{\Phi}$ or from its moments. Gautschi (1982) describes several methods. Root extraction is also extremely simple, since all roots are real, distinct, and located on $(\tau_2^{-1}, \tau_1^{-1})$, and roots of successive polynomials of the set interlace.

## 4 Numerical example

### 4.1 SPECIALIZATION TO CONSTANT-Q ABSORPTION BAND

As an example of the general method developed in the previous section, we treat the important special case in which $Q^{-1}$ is nearly constant over a frequency band. This is the most important case for seismic applications, and admits a closed-form solution for the poles and residues in terms of tabulated quantities.

Following Liu *et al.* (1976), we introduce a relaxation spectrum $\Phi(\ln \tau)$ which is constant in the interval $[\ln \tau_1, \ln \tau_2]$. This corresponds to constant $\hat{\Phi}$,

$$\hat{\Phi}(p) = \begin{cases} [\ln(\tau_2/\tau_1)]^{-1}, & \tau_2^{-1} \leq p \leq \tau_1^{-1} \\ 0, & p < \tau_2^{-1} \text{ or } p > \tau_1^{-1}, \end{cases} \tag{33}$$

in which we have taken account of the normalization $\phi(0)=1$. For a broad relaxation spectrum with $\tau_2^{-1} \ll \omega \ll \tau_1^{-1}$, $Q^{-1}$ is nearly frequency independent. If we also assume $\delta M/M_u \ll \ln(\tau_2/\tau_1)$, this nearly constant $Q^{-1}$ is given approximately by

$$Q^{-1} \sim \frac{\pi}{2}\frac{\delta M}{M_u}\left(\ln\frac{\tau_2}{\tau_1}\right)^{-1} = Q_0^{-1}.$$

The change of variables

$$x = \frac{2p-(\tau_1^{-1}+\tau_2^{-1})}{(\tau_1^{-1}-\tau_2^{-1})}$$

maps the distribution (33) on to the interval $[-1, 1]$. The Legendre polynomials $P_n$ are orthogonal with respect to unit weight on $[-1, 1]$. Therefore, the poles $\nu_i$ and residues $\lambda_i$ required for our $n$th-order approximant are given in terms of the zeros of the Legendre polynomials, $l_i$, by

$$\nu_i = \tfrac{1}{2}\,[l_i(\tau_1^{-1}-\tau_2^{-1})+(\tau_1^{-1}+\tau_2^{-1})]$$

and (from equation 24)

$$\lambda_i = \frac{k_n\, h_{n-1}\,(\tau_1^{-1}-\tau_2^{-1})}{2\,k_{n-1}\ln(\tau_2/\tau_1)\,P_{n-1}(l_i)\,P_n'(l_i)}$$

$$= \frac{(\tau_1^{-1}-\tau_2^{-1})}{n\ln(\tau_2/\tau_1)\,P_{n-1}(l_i)\,P_n'(l_i)}, \qquad i=1,\ldots n.$$

Here $k_n$ is the leading coefficient of $P_n$, and $h_n$ is its quadratic integral on $[-1, 1]$. Then, for this special case, (27) can be written as

$$\frac{d\zeta_i(t)}{dt}+\nu_i\,\zeta_i(t) = \left(\frac{\tau_1^{-1}-\tau_2^{-1}}{\pi}\,w_i\,Q_0^{-1}\right)\epsilon(t), \qquad i=1,\ldots n, \tag{34}$$

where

$$\nu_i = \tfrac{1}{2}\,[l_i(\tau_1^{-1}-\tau_2^{-1})+(\tau_1^{-1}+\tau_2^{-1})] \tag{35}$$

$$w_i = \frac{2}{n\,P_{n-1}(l_i)\,P_n'(l_i)}. \tag{36}$$

The $l_i$ and $w_i$ are, respectively, the abscissas and weights for $n$-point Gauss–Legendre quadrature, and their numerical values are extensively tabulated (e.g. Abramowitz & Stegun 1964, p. 916).

### 4.2 NUMERICAL SOLUTION

A suitable finite difference scheme for time-stepping each of the $n$ equations (34), with time step $\Delta t$, is the recursion

$$y_j = a y_{j-1} + b x_j$$

where

$$y_j = \zeta_i(j\Delta t)$$

$$x_j = \epsilon(j\Delta t) + \epsilon(j\Delta t - \Delta t)$$

$$a = \frac{2 - \Delta t v_i}{2 + \Delta t v_i}$$

$$b = \frac{\Delta t}{2 + \Delta t v_i} \frac{\tau_1^{-1} - \tau_2^{-1}}{\pi} w_i Q_0^{-1}.$$

This scheme can be readily incorporated into a finite difference method for the scalar wave equation. As a test, we apply the method to model propagation of an impulsive plane wave in a homogeneous medium. The width of the constant-$Q$ absorption band is given in our test problem by $\tau_2/\tau_1 = 10^4$, and $Q_0$ is equal to 20. The initial pulse $U_0(t)$ in this test calculation was given the form

$$U_0(t) = \begin{cases} \sin^2(\pi t/10\tau_1) & t/\tau_1 < 10 \\ 0 & t/\tau_1 > 10. \end{cases}$$

The time step was $\Delta t = 0.1\,\tau_1$, giving an initial pulse width of 100 $\Delta t$. This relatively broad, smooth input pulse is intended to minimize the usual numerical dispersion associated with the spatial discretization of the problem, which is unrelated to the differential approximation of the attenuation operator being tested here.

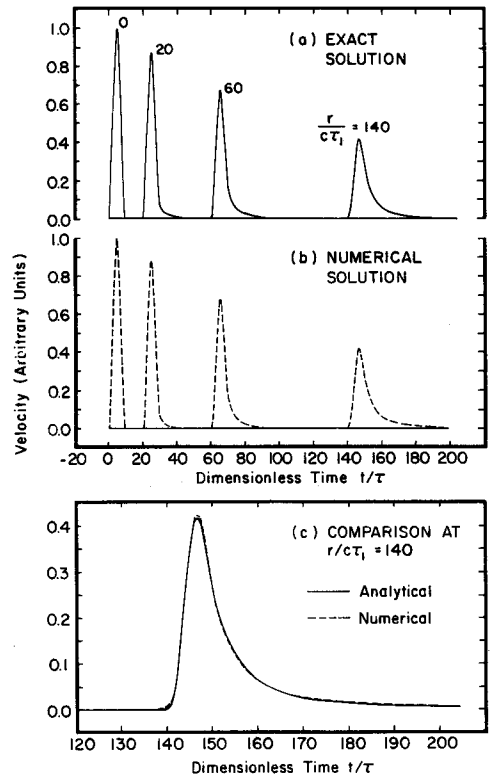


**Figure 1.** Comparison of: (a) exact solution and (b) approximate numerical solution to the constant-$Q$ test problem. Part (c) of the figure overlays the two solutions at the largest range, $r = 140\,c\tau_1$ ($\tau_1$ is the minimum relaxation time and $c$ is the wavespeed associated with the unrelaxed modulus), which is the worst case. The two solutions are virtually indistinguishable.

Fig. 1 shows the numerical results for plane wave propagation, obtained with the 5-pole approximant. This solution is compared with an exact solution for the absorption-band model obtained by Fourier synthesis. The initial pulse is shown, together with the propagated pulse at three different ranges. The worst case is at the largest range, and the exact and numerical solutions for this range are superimposed on an expanded scale in Fig. 1(c). Even in this case, the finite difference and exact solutions are nearly indistinguishable. Those differences which are visible, such as the small precursor in the finite difference solution, are attributable to the spatial differencing of the wave equation, rather than to the approximation of the attenuation operator.

## 5 Discussion

The preceding development solves a long-standing problem in the time-domain computation of synthetic seismograms, that of representing realistic attenuation laws accurately in a form suitable for time stepping. The numerical example exhibits the extraordinary accuracy of the method in the case of a constant-$Q$ attenuation law. Generalization of the 1-D treatment of this paper to higher dimensional, isotropic media is immediate, since the tensor stress–strain relationship of a linear, isotropic material can always be reduced to a set of separate scalar stress–strain relations of the form considered here.

Potential applications of the method include 2-D forward modelling of reflection seismograms and full wave acoustic logs. In both cases, time-marching methods are routinely used to simulate acoustic and elastic waves as they propagate over distances of several tens of wavelengths (e.g. Kelly *et al.* 1976, Kosloff & Baysal 1982 and Chang 1983). Anelastic losses may have pronounced effects on pulse amplitudes and dispersion over paths of this length. Indeed, a major objective of full wave acoustic logging is to deduce the anelastic properties of the path from the observed attenuation of microseismogram phases.

Another potential application is in earthquake strong motion modelling, where the DWFE method (Olson *et al.* 1984) has received much use. Anelastic effects can be expected to play a significant role at source–receiver ranges of tens of kilometres, especially if low-velocity, low $Q$ surface layers are present.

A further application of the method is for modelling the near-source transition from non-linear to linear attenuation. The laboratory results summarized by Mavko (1979) indicate that stress wave attenuation in many rock types is amplitude-dependent for strains above about $10^{-6}$. A corollary proposed by Trulio (1978) and analysed by Minster & Day (1984) is that non-linear attenuation in the near-field of underground nuclear explosions persists to ranges several times greater than the regime of large-scale rock failure (the so-called elastic radius). The importance of the transition from non-linear, amplitude-dependent attenuation to linear, anelastic attenuation is that the wavefield at the transition radius defines the equivalent forces, or seismic coupling, of the explosion. Numerical modelling of this phenomenon calls for a combined time domain treatment of non-linear and anelastic attenuation laws.

## 6 Conclusion

The following remarks summarize our conclusions from this study:

(1) The Padé approximant method permits realistic anelastic attenuation laws to be conveniently and accurately represented in a form suitable for time stepping.

(2) All approximants lead to $Q$ operators which are causal, stable, and dissipative, and which form uniformly convergent sequences.

(3) The case of constant $Q$ is especially convenient, in that all required coefficients have been derived in closed form.

(4) The method is apt to find applications in several areas of seismic forward modelling including synthesis of reflection seismograms, full wave acoustic logs, and earthquake strong motion, as well as in theoretical studies of non-linear source coupling.

## Acknowledgments

The authors thank Bill Farrell for suggesting several improvements to the manuscript. This research was supported by the Advanced Research Projects Agency of the Department of Defense, and was monitored by the Air Force Office of Scientific Research under Contract No. F49620-81-C-0094.

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## Appendix

To facilitate discussion of convergence, it will be useful to view equation (1) as defining a linear operator on elements $\epsilon(t)$ of the function space $L_\infty[0, T]$. The space $L_\infty$ is the Banach space of essentially bounded strain histories on the finite time interval $[0, T]$, with the norm given by the essential supremum, $\| \epsilon \| = \text{ess sup } \{ | \epsilon(t) | : 0 \leqslant t \leqslant T \}$. With the restrictions on $M(t)$ given by equations (5) and (6), equation (1) defines a bounded linear operator $\mathcal{M}$ from $L_\infty$ into itself (boundedness of $\mathcal{M}$ follows readily from the fact that $\phi(t)$ is absolutely integrable on $[0, T]$). With the introduction of the uniform operator norm, defined for the arbitrary linear operator $\mathcal{L}$ by

$$\| \mathcal{L} \| = \sup \left\{ \frac{\| \mathcal{L}\epsilon \|}{\| \epsilon \|} : \epsilon \neq 0 \right\}, \tag{A1}$$

the set of all bounded linear operators from $L_\infty$ into itself also constitutes a Banach space. Completeness of this space of operators is a consequence of the completeness of $L_\infty$.

The linear differential operator of order $n$ defined by (26) and (27) will be denoted by $\mathcal{M}_n$. $\mathcal{M}_n$ is bounded, since the roots $\nu_i$ all lie in the interior of the interval $[\tau_2^{-1}, \tau_1^{-1}]$ and $\tau_1 > 0$. We show that $\mathcal{M}_n$ converges uniformly to $\mathcal{M}$, i.e. in the sense of the uniform operator norm (A1).

From equations (1), (5) and the continuity of $\phi(t)$, it follows that

$$\lim_{n \to \infty} \int_0^T | \dot{\phi}(t) - \dot{\phi}_n(t) | \, dt = 0 \tag{A2}$$

is a sufficient condition for

$$\lim_{n \to \infty} \| \mathcal{M} - \mathcal{M}_n \| = 0.$$

In (A2), $\dot{\phi}_n$ is the relaxation-rate function associated with $\mathcal{M}_n$, obtained by Laplace inversion of equation (25) and use of equation (5). The integrand is given by

$$| \dot{\phi}(t) - \dot{\phi}_n(t) | = \int_{\tau_2^{-1}}^{\tau_1^{-1}} \exp(-pt) \, \hat{\Phi}(p) \, dp - \sum_{i=1}^{n} \lambda_i \exp(-\nu_i t). \tag{A3}$$

But in (A3) the summation term is a Gaussian quadrature formula for the integral term. Therefore, the right hand side is bounded by the Guassian quadrature error formula (equation 29), and we have

$$| \dot{\phi}(t) - \dot{\phi}_n(t) | \leqslant \frac{t^{2n} \exp(-t/\tau_2)}{(2n)! \, k_n^2}. \tag{A4}$$

The coefficient $k_n$ is the leading coefficient of the orthogonal polynomial $p_n$. Using the extremal property of orthonormal polynomials (Szegö 1939, p. 39), and noting that $\hat{\Phi}(p) = 0$ for $p < \tau_2^{-1}$, we conclude that $k_n^2 \geq \tau_1^{2n+1}$. Then (A4) becomes

$$|\dot{\phi}(t) - \dot{\phi}_n(t)| \leq \frac{(t/\tau_1)^{2n}}{(2n)!} \tau_1^{-1} \exp(-t/\tau_2), \tag{A5}$$

proving that $\{\dot{\phi}_n\}$ converges uniformly to $\dot{\phi}$ on any finite interval, therefore also establishing (A2).