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RMS Response of a One-Dimensional Half-space to *SH*

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Abstract We examine the extent to which the response of a perfectly elastic half-space to an *SH*-wave incident from below can be characterized when knowledge about the elastic structure is limited to the near surface. Elastic properties are modeled as piecewise continuous functions of the depth coordinate. It is found that the site amplification function can be determined with a frequency resolution that depends inversely on the depth to which the elastic structure is known. Specifically, certain spectral averages of the site amplification function, concentrated over bandwidth Δf, depend only on the elastic structure down to a two-way travel-time depth of $1/\Delta f$. These spectral averages are entirely independent of the elastic properties at greater depth. Equivalently, when the incident motion has a bandlimited white power spectrum of bandwidth Δf, the site amplification of the root mean square (rms) ground motion depends only on the elastic structure down to a two-way travel-time depth of $1/\Delta f$. When the bandwidth is sufficiently large, the following corollary applies: the rms surface ground motion equals the rms incident motion multiplied by $2\sqrt{I_b/I_0}$, where I_0 and I_b are shear impedances at the ground surface and basement depth, respectively. This result provides justification for a procedure conventionally used to correct stochastic estimates of earthquake ground motion to account for local site effects. The analysis also clarifies the limitations of that conventional procedure.

The results define specific site-response parameters that can be computed from knowledge of shallow structure alone and may thereby contribute to improved understanding of the physical basis for, and limitations of, site classification schemes that are based on average *S*-wave velocity at shallow depth. While the analytical results are rigorous only for infinite Q, numerical experiments indicate that similar results apply to models with finite, frequency-independent Q. The practical utility of the results is likely to be limited primarily by the degree of lateral heterogeneity present near sites of interest and the degree to which the sites respond nonlinearly to incident ground motion.

Introduction

Local site effects have an enormous influence on the intensity and character of earthquake ground motion. This has been recognized for many years and has been well illustrated in studies of several recent earthquakes (e.g., Singh *et al.*, 1988; Hough *et al.*, 1990; Borcherdt and Glassmoyer, 1992). A number of empirical and theoretical tools have been developed for assessing those site effects (see, e.g., Aki, 1988). The most advanced of these tools require detailed site characterization. For example, given a complete description of the seismic velocity structure in the three-dimensional volume surrounding a site, present computational methods (e.g., finite-difference and finite-element methods) are, in principle, capable of computing the site response to a given input ground motion. This approach is computationally very intensive by current standards, though it may be worthwhile undertaking for a few specific sites containing critical facilities, especially where there is already extensive knowledge of the local subsurface geology and prospective seismic sources. However, incompleteness of our geologic knowledge of the site ordinarily renders such computations impractical.

Moreover, applications such as the seismic zonation of a metropolitan area require that site-effect estimates be made with dense geographical coverage. In such applications, the scope of the problem is such that it may be infeasible to acquire geological information beyond a description of the surface geology and geotechnical characterization of the upper few tens of meters of the subsurface.

In light of this need for estimates of ground motion based on highly simplified geologic information, it is appropriate to reconsider highly simplified analytical models for site response. We focus on the case of horizontal stratification. Matrix methods for computing the response of horizontally layered viscoelastic models (e.g., Kennett, 1983) have been in use for more than 4 decades. One-dimensional models of this type have been subjected to rather detailed experimental tests (Cramer, 1995), and the sensitivity of site-response calculations to uncertainties in the model parameters has been studied numerically (Field and Jacob, 1993).

This article addresses the response of perfectly elastic, horizontally stratified surficial layering excited by a plane *SH*-wave incident from an underlying uniform half-space. Specifically, the shear modulus $\mu(z)$ and density $\rho(z)$ are assumed to be arbitrary piecewise continuous functions of depth z. To simplify the discussion, we will refer to the underlying uniform half-space as "basement," and we will assume in all cases that the basement shear impedance is known. Within the context of this idealization, we consider what information about site response can be determined when the only additional information available is the elastic structure at shallow depth, i.e., when $\mu(z)$ and $\rho(z)$ are known only for z less than some depth h.

We construct spectral averages of the site amplification (relative to basement), with averaging bandwidth Δf, which depend only on the elastic structure down to a two-way travel-time depth of $1/\Delta f$. Thus, from near-surface structure alone, even with complete ignorance of structure at greater depth, we can make some precise statements about the spectral response of the site. Then we define the rms amplification, $\overline{G^2_{\Delta f}}^{1/2}$, as the rms response at the site when the incident motion is a stationary stochastic process with unit rms amplitude and a bandlimited white power spectrum of width Δf. This quantity is of particular significance because bandlimited white noise (BLWN) has become a standard analytical model for the prediction of earthquake ground acceleration (e.g., Hanks and McGuire, 1981; Boore, 1983). In the BLWN model, the expected value of peak ground acceleration is directly proportional to rms acceleration and therefore to $\overline{G^2_{\Delta f}}^{1/2}$. The rms amplification is also shown to depend only on the elastic structure down to a two-way travel-time depth of $1/\Delta f$.

A special case of the rms result is of interest. Even if the only site information available is the shear impedance of the surface material, the rms amplification can still be determined, provided the input bandwidth is sufficiently large. In that case, $\overline{G^2_{\Delta f}}^{1/2}$ is given by

$$\overline{G^2_{\Delta f}}^{1/2} = 2\sqrt{\frac{I_b}{I_0}},$$

where I_0 and I_b are shear impedances at the ground surface and basement depth, respectively. For transient instead of stationary input, the square of the above expression gives the site amplification of the total signal energy.

The Site Amplification Function

We consider *SH* motion of a perfectly elastic half-space. The half-space has a one-dimensional structure; i.e., shear modulus and density are piecewise continuous, real functions $\mu(z)$ and $\rho(z)$, respectively, of the depth coordinate z only, with $z \geqq 0$. The shear velocity $\sqrt{\mu/\rho}$ is denoted by $\beta(z)$. The structure is terminated above by a free surface (at $z = 0$) and merges below into a uniform half-space; that is, there is a z_b (the "basement depth") such that the functions $\beta(z)$ and $\rho(z)$ are equal to constants β_b and ρ_b, respectively, for all $z > z_b$. The quantity of interest is the surface displacement excited by a plane *SH*-wave incident from below, with horizontal slowness p, and the restriction $p^{-1} > \sup\beta(z)$. The latter stipulation insures subcritical incidence; i.e., the horizontal phase velocity is bounded below by the maximum value of the shear velocity. We define the site impulse response $g(t)$ as the surface displacement induced by an incident wave of the form $\delta(t - px + \eta z)$, where $\eta = (\beta^{-2} - p^2)^{1/2}$. That is, denoting the induced displacement field by $u_\delta(z, t - px)$, the impulse response is

$$g(t) \equiv u_\delta(0, t). \tag{1}$$

To simplify the notation, we do not write out explicitly the dependence on slowness p. The corresponding site transfer function $G(f)$ is the Fourier transform of $g(t)$. More generally, where the incident wave time series is a stationary stochastic process u_{inc}, with power spectrum $S_{u_{\text{inc}}}$, and the corresponding surface motion time series is u_0, with power spectrum S_{u_0}, $|G(f)|^2$ can be interpreted as the spectral amplification function of the site:

$$|G(f)|^2 = \frac{S_{u0}(f)}{S_{u_{\text{inc}}}(f)}. \tag{2}$$

After Fourier transformation on time, the equation of motion for the impulse response field, for slowness p, is

$$(2\pi f)^2\mu\eta^2 U_\delta - \frac{\partial}{\partial z}\left(\mu\,\frac{\partial U_\delta}{\partial z}\right) = 0. \tag{3}$$

The transformed displacement $U_\delta(z, f)$ also satisfies the free surface condition at $z = 0$,

$$\mu\,\frac{\partial U_\delta}{\partial z} = 0, \quad z = 0. \tag{4}$$

Since the half-space is uniform beneath basement depth, we can express U_δ as the sum of the incident field and a downgoing reflected field U_δ^D:

$$U_\delta = e^{i2\pi f\eta z} + U_\delta^D(z_b, f)e^{-i2\pi f\eta z}, \quad z \geqq z_b. \tag{5}$$

We now derive a reciprocal relationship between the foregoing "site response" and the "site radiation" $u_r(z,t - px)$, i.e., the excitation of the half-space by an applied surface traction $\tau_0(t - px)$. The transformed displacement field satisfies the equation of motion,

$$(2\pi f)^2\mu\eta^2 U_r - \frac{\partial}{\partial z}\left(\mu\frac{\partial U_r}{\partial z}\right) = 0, \tag{6}$$

together with the radiation condition, which implies

$$U_r(z, f) = U_r(z_b, f)e^{-i2\pi f\eta z}, \quad z \geqq z_b. \tag{7}$$

The boundary condition is

$$\mu\frac{\partial U_r}{\partial z} = -T_0(f), \quad z = 0, \tag{8}$$

where $T_0(f)$ is the Fourier transform of $\tau_0(t)$. Multiplying (3) by U_r and (6) by U_δ, subtracting, and integrating by parts from 0 to z_b yields

$$U_\delta\mu\frac{\partial U_r}{\partial z}\bigg|_0^{z_b} = U_r\mu\frac{\partial U_\delta}{\partial z}\bigg|_0^{z_b}. \tag{9}$$

Then, inserting the conditions (4), (5), (7), and (8) into (9), and using definition (1), we obtain the following expression for the site transfer function in terms of the site radiation:

$$G(f) = 2\frac{2\pi i f I_b U_r(z_b, f)}{T_0(f)}, \tag{10}$$

where I_b is the basement impedance,

$$I_b = \rho_b\beta_b^2(\beta_b^{-2} - p^2)^{1/2}. \tag{11}$$

Taking the squared absolute value of both sides of (10) yields

$$|G(f)|^2 = 4I_b\frac{(2\pi f)^2 I_b U_r(z_b, f)U_r^*(z_b, f)}{T_0(f)T_0^*(f)}. \tag{12}$$

Equation 12 expresses the site amplification function (for slowness p) in terms of solutions to the reciprocal radiation problem (for the same slowness p). For the special case of uniform, constant-travel-time layers, a result equivalent to (12) has been derived using layer matrix methods (Scherbaum, 1987), by analogy with a corresponding relationship for acoustic waves (Claerbout, 1976).

Spectral Averages and RMS Amplification

We use (12) to show that certain spectral averages of the site amplification function depend only on shallow elastic structure. Multiplying (12) by $T_0T_0^*$ and integrating, we obtain

$$\int_{-\infty}^{\infty} |G(f)|^2 T_0(f)T_0^*(f)\,df = 4I_b\int_{-\infty}^{\infty}(2\pi f)^2 I_b U_r(z_b, f)U_r^*(z_b, f)\,df. \tag{13}$$

The right-hand integral in (13) is the energy (per unit area) radiated into the half-space $z > z_b$, which must equal the work done (per unit area) on the free surface by traction $\tau_0(t)$. The energy equality is insured by our requirement $p^{-1} > \sup\beta(z)$. Thus (13) can be written as

$$\int_{-\infty}^{\infty} |G(f)|^2\, T_0(f)T_0^*(f)\,df = 4I_b\int_{-\infty}^{\infty}\dot{u}_r(0, t)\tau_0(t)\,dt. \tag{14}$$

In (14), we can view $T_0T_0^*$ as playing the role of a spectral averaging kernel. This equation thus provides an expression for spectral averages of $|G|^2$ which only involves the transient power input at the free surface. Therefore, we can isolate the effects of the shallow part of the structure by time windowing the power input through our specification of the surface traction $\tau_0(t)$.

Alternately setting $\tau_0(t)$ to $b(t)\cos(2\pi f_0 t)$ and $b(t)\sin(2\pi f_0 t)$ in (14) and adding, we obtain

$$\int_{-\infty}^{\infty} |G(f)|^2\, V(f - f_0)\,df = 4I_b\int_0^{\infty} y(t)v(t)\cos(2\pi f_0 t)\,dt, \tag{15}$$

where $v(t)$ is the autocorrelation of $b(t)$, $V(f)$ is its spectrum, and $y(t - px)$ is the surface response to impulsive surface traction $\delta(t - px)$; i.e.,

$$\dot{u}_r(0, t) = \int_0^t y(t')\tau_0(t - t')\,dt'. \tag{16}$$

The Fourier transform $Y(f)$ of $y(t)$ can be thought of as the free-surface admittance function (for slowness p).

Finally, let the autocorrelation $v(t)$ equal $w(\Delta f t)$, where the normalized autocorrelation $w(t)$ has the properties

$$\begin{aligned} w(t) &= 0 \quad \text{for } |t| > 1, \\ w(0) &= 1, \\ \int_{-1}^{1} w(t)\,dt &= 1. \end{aligned} \tag{17}$$

That is, the power input is modulated by a function $b(t)$, which has an autocorrelation $w(\Delta f t)$ with support interval $[-\Delta f^{-1}, \Delta f^{-1}]$ and a spectrum with unit area and equivalent width (area divided by zero-frequency ordinate) Δf. Equation (15) then becomes

$$\frac{1}{\Delta f}\int_{-\infty}^{\infty} |G(f)|^2\, W\!\left(\frac{f-f_0}{\Delta f}\right) df = 4I_b \int_0^{1/\Delta f} y(t)w(\Delta f t)\cos(2\pi f_0 t)\, dt, \quad (18)$$

where $W(f)$ is the Fourier transform of $w(t)$. The left-hand side of (18) is just a weighted average of the site amplification spectrum, defined with respect to the averaging kernel W, centered at f_0, with equivalent width proportional to Δf. Because of the windowing supplied by w, the right-hand side of (18) depends only on $y(t)$ for $t < 1/\Delta f$, and we have therefore reduced the upper integration limit of (15) accordingly. Then $y(t)$, and therefore (18), depends only on the earth structure on the interval $0 \leqq z \leqq h(1/\Delta f)$, where $h(s)$ is the depth corresponding to two-way vertical travel time s,

$$s(h) = 2\int_0^h \eta(z)\, dz. \quad (19)$$

We denote the spectral averages in (18) by $\overline{G^2_{\Delta f}}(f_0)$:

$$\overline{G^2_{\Delta f}}(f_0) \equiv \frac{1}{\Delta f}\int_{-\infty}^{\infty} |G(f)|^2\, W\!\left(\frac{f-f_0}{\Delta f}\right) df. \quad (20)$$

Equation (18) provides an exact expression for spectral averages of the site amplification when the averages are defined with respect to any spectral averaging kernel W derived from an autocorrelation function w with support interval $[-1, 1]$. Since $\overline{G^2_{\Delta f}}(f_0)$ in (18) depends only on elastic properties above $h(1/\Delta f)$, we have proved that all models that have the same elastic properties above this level will share the same spectral averages of form (20). Equation (18) applies to any value of the central frequency f_0. Since Δf is the equivalent width of the averaging kernel, it is appropriate to call $\overline{G^2_{\Delta f}}$ the spectral average of the site amplification over bandwidth Δf.

An equivalent statement of result (18) is the following: if the input ground motion u_{inc} is a stationary process with power spectral shape W, central frequency f_0, and bandwidth Δf, i.e.

$$S_{u_{\text{inc}}}(f) = W\!\left(\frac{f-f_0}{\Delta f}\right), \quad (21)$$

then the rms incident motion is amplified by factor $\overline{G^2_{\Delta f}}^{\,1/2}$ to give rms surface motion:

$$\text{RMS Amplification} \equiv \frac{\sqrt{E[u_0^2(t)]}}{\sqrt{E[u_{\text{inc}}^2(t)]}} = \sqrt{\overline{G^2_{\Delta f}}(f_0)}. \quad (22)$$

This result follows from the equality between the zero-lag autocorrelation and the integral of the corresponding power spectrum. Naturally, such rms amplification factors can be defined with respect to input acceleration or velocity as well, just by specifying input acceleration or velocity in place of input displacement in (21). Each such rms amplification factor, for input with a given bandwidth Δf and central frequency f_0, is the same for all models that share the same elastic properties above level $h(1/\Delta f)$. A similar formulation applies to transient input with energy spectrum W; in that case, the total spectral energy in the ground motion will be amplified by $\overline{G^2_{\Delta f}}$.

This rms viewpoint may prove useful because of the extensive use that is made of spectral models (e.g., the Brune (1970) spectral model) to predict rms ground acceleration from earthquakes, following Hanks and McGuire (1981). Earthquake ground motion is transient rather than stationary, but for large earthquakes, the duration of the incident transient will usually be large compared with the duration of the site impulse response. In that case, the rms of the incident acceleration will be amplified by approximately $\overline{G^2_{\Delta f}}^{\,1/2}$ to give the rms of the surface acceleration. Furthermore, the expected value of peak ground acceleration, as estimated via random vibration theory, is approximately proportional to the rms acceleration, with relatively weak dependence on duration and spectral shape (see, e.g., Boore, 1983; Udwadia and Trifunac, 1974). Therefore, the rms amplification is expected to be a good approximation to the site amplification factor for peak acceleration.

The above results are exact only for perfect elasticity, i.e., infinite Q. Anderson *et al.* (1996) have used numerical experiments to extend the analysis to models with finite, frequency-independent Q. They find that the rms amplification is still approximately independent of shear velocity beneath the near-surface region, in the manner predicted by the infinite Q analysis given above, provided variations in the model preserve a fixed value for the loss exponent t^*. Anderson *et al.* show that the rms amplification may be quite sensitive to that t^* value, however. In this context, t^* is defined as the travel-time weighted integral of Q^{-1} taken over the entire shear-wave ray path.

The applicability of the foregoing results will also be limited in the presence of strong lateral heterogeneity. In alluvial basin deposits, for example, horizontal interference patterns will be excited in addition to the vertical interference patterns present in one-dimensional models (see, e.g., Wong and Trifunac, 1974). In such cases, rms amplification estimates based on any one-dimensional framework can at best be considered first-order approximations.

Numerical Example

A specification for $w(t)$ that meets the bounded support criterion in (17) is the triangular window $w(t) = 1 - |t|$. In this case, the averaging kernel, denoted by W_S, is the squared sinc function

$$W_S(f) = \sin^2(\pi f)/(\pi f)^2. \tag{23}$$

An example of spectral averaging over this kernel is shown in Figure 1. Figure 1a shows two Earth models that coincide down to a depth of 50 m, corresponding to a two-way travel time of 0.45 sec. Beneath that depth, the models diverge (but share a common basement impedance). The broken curves in Figure 1b show the corresponding transfer function amplitudes $|G|$, for the case of vertical incidence. The solid curve in Figure 1b is $\overline{G^2_{\Delta f}}^{\,1/2}$ for an averaging bandwidth of 2.5 Hz, plotted as a function of central frequency f_0. As required by equation (18), $\overline{G^2_{\Delta f}}^{\,1/2}$ is identical for the two models.

In practice, we find that the bounded support criterion can usually be relaxed somewhat without significantly compromising the independence of spectral averages on structure beneath $h(1/\Delta f)$. Figure 2 compares rms averages of the two models in Figure 1a, as a function of central frequency, for a 2.5-Hz bandwidth. In this case, we have already verified that the averaging kernel W_S gives identical results for the two models. Also shown in Figure 2 are the rms results for both Gaussian and Lorentzian averaging kernels, W_G and W_L,

$$W_G(f) = e^{-\pi f^2}, \tag{24}$$

$$W_L(f) = \frac{1}{1 + (\pi f)^2}. \tag{25}$$

Averages over the Gaussian kernel show only negligible differences between the models, even though this kernel does not strictly correspond to an autocorrelation of bounded support. Likewise, averages over the Lorentzian kernel show only small differences between the two models.

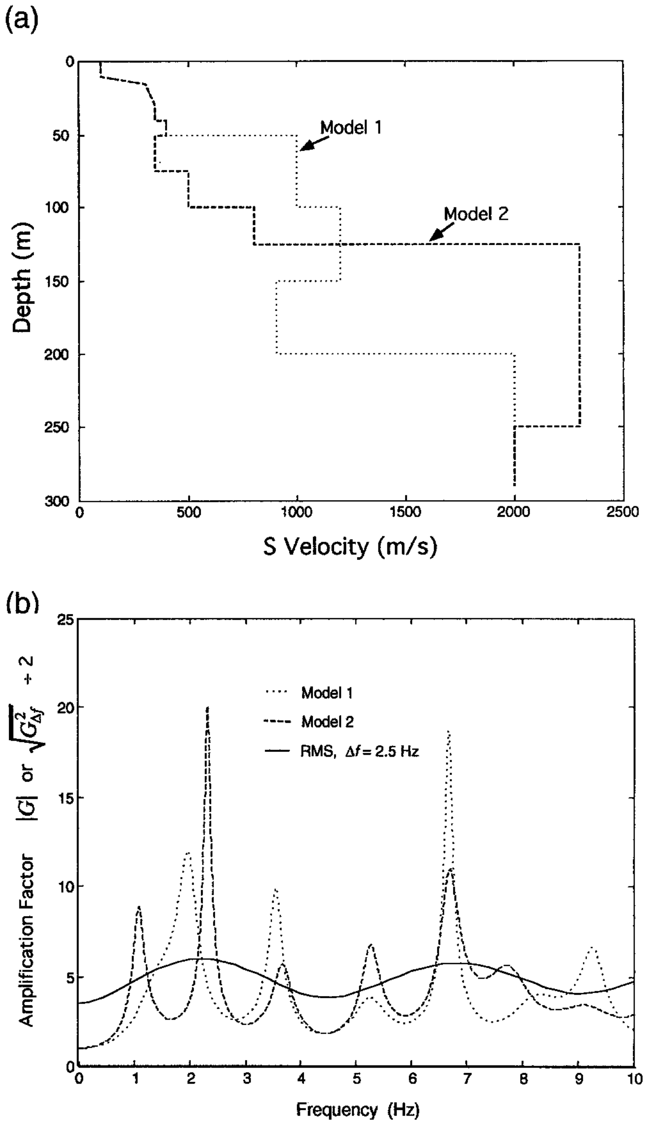

Figure 1. (a) Two models of S velocity versus depth. The S-velocity profiles of the two models coincide down to a depth of 50 m, corresponding to a two-way travel time of 0.45 sec. In both cases, model density is 1800 kg/m^3 above 50 m and 2200 kg/m^3 below 50 m. (b) Broken curves show the transfer function amplitude $|G|$ for each of the two models, assuming vertically incident *SH*. The solid curve is the rms amplification, for input bandwidth, Δf, equal to 2.5 Hz. The rms amplification is plotted as a function of central frequency of the input, f_0, for input spectral shape W_S (i.e., the sinc squared kernel). As predicted by equation (18), the rms amplification is identical for the two models.

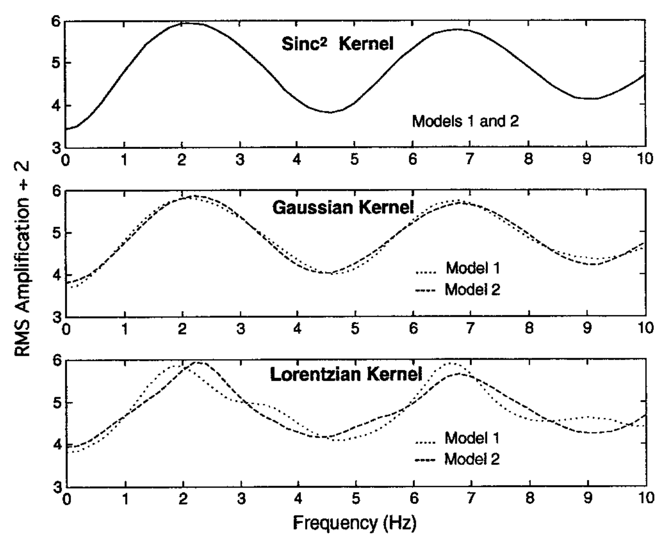

Figure 2. Comparison of rms amplification for the two models in Figure 1a. The rms amplification is plotted versus central frequency f_0, for an input bandwidth of 2.5 Hz. The comparison is made for the three averaging kernels W_S, W_G, and W_L. Since W_S conforms to the bounded support criterion (17), it yields identical averages for the two models. Both the Gaussian and Lorentzian kernels, although not strictly in conformance with (17), show only small difference in rms amplification for the two models. Thus, in practice, the bounded support criterion may be relaxed somewhat without significantly compromising the independence of spectral averages on deep structure.

Applications

RMS Response, Known Surface Layer. A consequence of Equation (18) is that even if we know only the shear impedance of a thin surficial layer, we can determine the rms amplification exactly, provided the spectrum of the incident wave is sufficiently broad. The shear velocity and impedance of a surface layer of depth H_0 will be denoted by β_0 and I_0, respectively, where $I_0 = \rho_0\beta_0^2(\beta_0^{-2} - p^2)^{1/2}$. We take

$$\Delta f \geqq \frac{1}{2H_0(\beta_0^{-2} - p^2)^{1/2}}, \tag{26}$$

in which case no reflected signal influences the surface admittance function $y(t)$ during the period $1/\Delta f$. Thus, we may use the uniform half-space admittance for $y(t)$:

$$y(t)w(\Delta ft) = I_0^{-1}\delta(t), \tag{27}$$

so the spectral average (18) becomes

$$\overline{G^2_{\Delta f}} = 4\frac{I_b}{I_0}, \tag{28}$$

and the rms amplification is

$$\overline{G^2_{\Delta f}}^{1/2} = 2\left(\frac{I_b}{I_0}\right)^{1/2}. \tag{29}$$

The factor of 2 in the rms amplification (29) is just the free-surface amplification of a uniform half-space. Interestingly, the remaining factor in (29) is identical to the plane wave amplification factor under conditions where the WKB approximation applies. That is, (29) coincides with the site transfer function one would obtain in the limit when the subsurface velocity profile is continuous and the frequency is restricted such that wavelength is everywhere short compared with the reciprocal of the gradient of the log of velocity. Thus, we have shown that the smooth-model WKB approximation to the *site transfer function* is also an exact expression for the *rms amplification,* irrespective of the actual complexity of the stratification and irrespective of the central frequency. The only limitations are that there be a uniform layer of finite thickness H_0 present at the surface, the input bandwidth be at least $\Delta f \geqq 1/2H_0(\beta_0^{-2} - p^2)^{1/2}$, and the incidence angle be small enough that the restriction $p^{-1} > \sup\beta(z)$ on the slowness is honored.

Standard spectral models for earthquake acceleration are approximately flat between a corner frequency f_c and a characteristic high-frequency limit $f_{\max}$ (Brune, 1970; Hanks, 1982). Hence, an extensively used earthquake acceleration model is BLWN, with passband $[f_c, f_{\max}]$. The foregoing analysis justifies the procedure (e.g., Boore, 1983) of using the WKB amplification factor to account for site amplification of the rms acceleration estimate obtained from these BLWN models. In the case of vertical incidence, for example, the procedure is justified provided the impedance I_0 characterizes a surface layer of thickness $\beta/2(f_{\max} - f_c)$ or greater. If the BLWN specification is relaxed slightly, so as to accommodate a time-limited autocorrelation, as in the first of criteria (17), then the procedure is exact.

Spectral Averages, Two Known Surface Layers. Next we consider the case in which shear velocity and impedance are known for two uniform surficial layers. To simplify the presentation, we treat the case of vertical incidence, since modifications required for $p > 0$ are obvious. The thickness H_0 of the upper layer is known, and the thickness of the underlying layer is known to equal or exceed H_1, beneath which the elastic properties are unknown. Layer velocities and impedances are denoted by the same subscripts as are the layer thicknesses.

For input bandwidth equal to or greater than $\beta/(2H_0)$, of course, the spectral average and rms results are identical to the single-layer case, as given by (28) and (29). However, knowing the properties of the second layer, we can now construct spectral averages for narrower input bandwidth as well. Bandwidth is now limited by the two-way travel time to depth $H_0 + H_1$,

$$\Delta f \geq \frac{1}{2}\left(\frac{\beta_0\beta_1}{H_0\beta_1 + H_1\beta_0}\right). \tag{30}$$

We can evaluate the spectral averages using (18), with the admittance $y(t)$ for the actual model replaced by that of a single layer over a uniform half-space of impedance I_1:

$$y(t) = I_0^{-1}[\delta(t) + 2r\delta(t - \tau) + 2r^2\delta(t - 2\tau)\ldots], \tag{31}$$

where r is the reflection coefficient at the bottom of the layer and τ is the two-way travel time through the layer,

$$r = \frac{I_0 - I_1}{I_0 + I_1}, \quad \tau = \frac{2H_0}{\beta_0}. \tag{32}$$

Performing the cosine transform in (18) then gives

$$\overline{G^2_{\Delta f}}(f_0) = 4\frac{I_b}{I_0}\int_{-\infty}^{\infty} \frac{1 - r^2}{1 + r^2 - 2r\cos(4\pi H_0 f/\beta_0)}\, W\left(\frac{f - f_0}{\Delta f}\right) df, \tag{33}$$

independent of the model structure beneath $H_0 + H_1$, so long as Δf conforms with (30).

RMS Response to BLWN, Two Known Surface Layers. Equation (33) can be applied to estimate the rms response to white noise with upper frequency limit $f_{\max}$, in the presence of two known surface layers. To represent BLWN with low-frequency cutoff 0 and upper cutoff $f_{\max}$, we set the center frequency f_0 equal to zero and the spectral half-width $\Delta f/2$ equal to $f_{\max}$ for the computation of $\overline{G^2_{\Delta f}}^{1/2}$. It then proves convenient to construct a reduced rms amplification function R,

$$R(\Omega) = \left(\sqrt{\frac{I_1}{I_0}} - 1\right)^{-1} \left(\sqrt{\frac{I_1}{4I_b} \overline{G^2_{2f_{\max}}(0)}} - 1\right). \quad (34)$$

The reduced rms amplification function is useful because it depends on $f_{\max}$ only through the dimensionless ratio $\Omega = H_0 f_{\max}/\beta_0$, is only weakly dependent on the impedance ratio I_1/I_0, and is independent of the remainder of the model as long as $f_{\max}$ exceeds $0.25\beta_0\beta_1/(H_0\beta_1 + H_1\beta_0)$, as stipulated by (30). Figure 3 shows $R(\Omega)$, as obtained from (33) using the kernel W_S, for various values of the impedance ratio I_1/I_0. Apart from oscillations near $\Omega = 0.1$ (artifacts of the sidelobes of the kernel function), $R(\Omega)$ shows little dependence on I_1/I_0 and for most practical purposes can probably be adequately approximated by a straight line of slope 4 through the origin, truncated at $\Omega = 0.25$. For all Ω above 0.25, (26) is satisfied, and therefore $R(\Omega)$ has unit value, as required by the single-layer result (29).

A convenient approximation can be derived by evaluating (33) using the Lorentzian kernel, in which case a closed form solution is obtained for $\overline{G^2_{\Delta f}(f_0)}$, though strict independence on the model structure beneath $H_0 + H_1$ is sacrificed. The result is

$$\overline{G^2_{\Delta f}(f_0)} \approx 4\frac{I_b}{I_0}\left[\frac{1 - r^2e^{-8H_0\Delta f/\beta_0}}{1 - 2re^{-4H_0\Delta f/\beta_0}\cos(4\pi H_0\Delta f/\beta_0) + r^2e^{-8H_0\Delta f/\beta_0}}\right]. \quad (35)$$

The integration of (33) leading to (35) is exact, but approximate equality is indicated in (35) to emphasize that the result is now only approximately independent of structure beneath $H_0 + H_1$. The corresponding approximation for $R(\Omega)$ is

$$R(\Omega) \approx \left(\sqrt{\frac{I_1}{I_0}} - 1\right)^{-1} \left\{\sqrt{\frac{I_1[1 - (re^{-8\Omega})^2]}{I_0(1 - re^{-8\Omega})^2}} - 1\right\}. \quad (36)$$

This closed-form expression has the added convenience of having negligible dependence on the impedance ratios, as shown in Figure 3.

The result that R is nearly independent of I_1/I_0 implies that the rms response in this case, $\overline{G^2_{2f_{\max}}(0)}^{1/2}$, is a weighted sum of $(I_0)^{-1/2}$ and $(I_1)^{-1/2}$. Approximating R by the straight line of slope 4, for example, we obtain the approximation

$$\overline{G^2_{2f_{\max}}(0)}^{1/2} \approx 2\sqrt{I_b}\left[a\frac{1}{\sqrt{I_0}} + (1 - a)\frac{1}{\sqrt{I_1}}\right], \quad (37)$$

where the weighing factor a equals the ratio of $f_{\max}$ to the lowest resonance frequency of the top layer (or 1, if this ratio exceeds 1); i.e.,

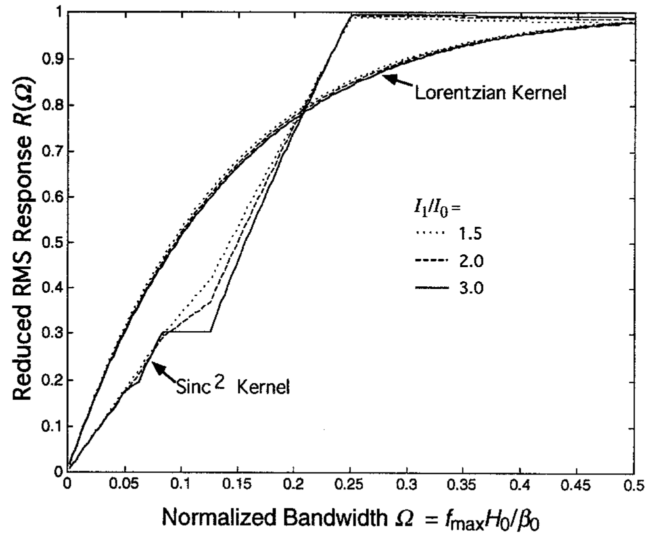

Figure 3. The reduced rms response to BLWN with upper cutoff frequency $f_{\max}$, in the presence of two surface layers with impedance contrast I_1/I_0. The reduced response is related to the rms amplification through equation (34). It depends on $f_{\max}$ only through the dimensionless ratio $\Omega = H_0 f_{\max}/\beta_0$ and is independent of the remainder of the model beneath a two-way travel-time depth of $1/(2f_{\max})$. As can be seen in the figure, the reduced rms response is only weakly dependent on the impedance ratio of the layers (although the full rms response is strongly dependent on the impedance ratio). Also plotted is the reduced rms response for the Lorentzian kernel, which, however, is only approximately independent of the deep portion of the model.

$$a = \min(4f_{\max}H_0/\beta_0, 1). \quad (38)$$

The interpretation of (37) and (38) is that the rms amplification in this case assumes the same form as it did in the single-layer case, (29), but with $(I_0)^{-1/2}$ replaced by an average of $(I_0)^{-1/2}$ and $(I_1)^{-1/2}$, travel-time-weighted, down to a one-way travel time depth of $1/4f_{\max}$, i.e., 1/4 of the cutoff period of the input spectrum. This result may provide some theoretical foundation for proposed site classification schemes that are based on average S-wave velocity at shallow depth, in which the average typically is taken over the uppermost 30 m (Boore *et al.*, 1993). The result may also provide some guidance on the use of travel-time-weighted average shear velocity as a predictor variable in regression analysis of strong motion, as in Boore *et al.* (1994). A complete analysis of these questions would have to take finite Q into account, however.

Summary

We have examined the extent to which the response to *SH* waves of a perfectly elastic, one-dimensional half-space can be characterized from partial information about its elastic structure. It has been shown that spectral averages of bandwidth Δf can be constructed that depend only on the elastic structure down to a two-way travel-time depth of $1/\Delta f$. Equivalently, when the incident wave has a bandlimited white power spectrum of bandwidth Δf, the site amplification of the rms ground motion depends only on the elastic structure down to a two-way travel-time depth of $1/\Delta f$. A corollary is that, provided the incident wave has sufficiently large bandwidth (as stipulated by equation 26), the rms incident ground motion will be amplified by twice the square root of the ratio (basement impedance $\div$ surface impedance).

To construct exact spectral averages of the site amplification function when the near-surface structure is known to some two-way travel-time depth $1/\Delta f$, model elastic properties can be assigned freely at depths greater than $1/\Delta f$, letting them assume any values convenient for computation. Then standard methods can be used to compute the site-response function for that specific structure, and the spectral averages of bandwidth Δf will be exact, irrespective of the actual deep structure. This approach was used to develop a relatively simple expression for the rms amplification in the case in which the elastic properties of two surficial layers are known. That expression was in turn used to generate numerical results for the rms amplification of white noise with upper cutoff frequency $f_{\max}$. The latter results can be expressed in a reduced form that seems to be nearly independent of the impedance ratio of the two surficial layers. As a consequence, the rms response can be well approximated by a weighted sum of the inverse square roots of the layer impedances, $(I_0)^{-1/2}$ and $(I_1)^{-1/2}$.

While the analytical results are rigorous only for infinite Q, numerical experiments indicate that similar results apply in an approximate sense to models with finite, frequency-independent Q (Anderson *et al.,* 1996). In that case, however, the Q structure throughout the model must be considered, through its effect on the loss exponent t^*.

The foregoing analysis, despite being built on a quite idealized theoretical framework, may offer some insights useful in microzonation studies, which by their nature must rely on rather minimal geological and geotechnical characterizations of the region of interest. In particular, the results may contribute to improved understanding of the physical basis for, and limitations of, site-classification schemes that are based on average S-wave velocity at shallow depth. The practical utility of the results will probably be limited primarily by the degree of lateral heterogeneity present near a site of interest and the degree to which the site responds nonlinearly to the incident ground motion.

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